GROTHENDIECK'S THEOREM FOR ABSOLUTELY SUMMING MULTILINEAR OPERATORS IS OPTIMAL

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Abstract. Grothendieck's theorem asserts that every continuous linear operator from ℓ_1 to ℓ_2 is absolutely $(1;1)$-summing. In this note we prove that the optimal constant g_m so that every continuous m-linear operator from $\ell_1 \times \cdots \times \ell_1$ to ℓ_2 is absolutely $(g_m;1)$-summing is $\frac{2}{m+1}$. We also show that if $g_m < \frac{2}{m+1}$ there is $\mathfrak{c}$ dimensional linear space composed by continuous non absolutely $(g_m;1)$-summing m-linear operators from $\ell_1 \times \cdots \times \ell_1$ to ℓ_2. In particular, our result solves (in the positive) a conjecture posed by A.T. Bernardino in 2011.

1. Introduction

A celebrated result of Grothendieck asserts that every continuous linear operator from ℓ_1 to ℓ_2 is absolutely $(1;1)$-summing. It was recently proved [3] that this result can be lifted to multilinear operators in the following fashion:

Every continuous m-linear operator from $\ell_1 \times \cdots \times \ell_1$ to ℓ_2 is absolutely $\left(\frac{2}{m+1};1\right)$-summing.

In the same paper the author conjectured that the value $\frac{2}{m+1}$ is optimal. A particular case of our main result gives a positive solution to this conjecture:

Theorem 1.1. *The estimate $\frac{2}{m+1}$ is optimal. Moreover, if $g_m < \frac{2}{m+1}$ then there exists a $\mathfrak{c}$-dimensional linear space formed (except by the null vector) by continuous non absolutely $(g_m;1)$-summing m-linear operators. This result is optimal in terms of dimension.*

Above, $\mathfrak{c}$ denotes the cardinality of the continuum. In other words, our main result shows that if $g_m < \frac{2}{m+1}$, the set of continuous non absolutely $(g_m;1)$-summing multilinear operators is $\mathfrak{c}$-lineable and moreover, maximal lineable. For the theory of lineability we refer to [1, 2] and the references therein.

Our proof of the optimality of $\frac{2}{m+1}$ is inspired on ideas that date back to the classical work of Lindenstrauss and Pełczyński [9] and, later, explored in a series of papers (see, e.g., [4, 5, 6, 10]).

Throughout this note, X and Y shall stand for Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The closed unit ball of X is denoted by B_X and the topological dual of X by X^*. Also, recall that a continuous linear operator $u : X \to Y$ is absolutely $(q;1)$-summing (see [7]) if there exists $C \geq 0$ such that

$$\left(\sum_{j=1}^{n} \|u(x_j)\|^q\right)^{\frac{1}{q}} \leq C \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{n} |\varphi(x_j)|$$

for every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$. The nonlinear theory of absolutely summing operators was designed by Pietsch in 1983 ([11]) and since then has been intensively studied. One of the possible polynomial generalizations of absolutely summing operators is the concept of absolutely summing polynomial. The space of continuous m-homogeneous polynomials from X to Y will be henceforth denoted by $\mathcal{P}(^m X; Y)$. Given a positive integer m and $q \geq \frac{1}{m}$, a continuous m-homogeneous polynomial $P : X \to Y$ is absolutely $(q;1)$-summing if there exists a constant

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$C \geq 0$ such that

$$\left(\sum_{j=1}^{n}\|P(x_j)\|^q\right)^{\frac{1}{q}} \leq C\left(\sup_{\varphi\in B_{X^*}}\sum_{j=1}^{n}|\varphi(x_j)|\right)^m$$

for every $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in X$. If $q<1$, the infimum of the constants C satisfying the above inequality is a Banach quasinorm for the space of absolutely $(q,1)$-summing polynomials from X to Y, and it is denoted by $\pi_{q,1}$. For multilinear mappings the definition is similar:

A continuous m-linear operator $T : X\times\cdots\times X \to Y$ is absolutely $(q;1)$-summing (with $q \geq \frac{1}{m}$) if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{n}\|T(x_j^{(1)},...,x_j^{(m)})\|^q\right)^{\frac{1}{q}} \leq C\prod_{k=1}^{m}\left(\sup_{\varphi\in B_{X^*}}\sum_{j=1}^{n}|\varphi(x_j^{(k)})|\right)$$

for every $n\in\mathbb{N}$ and $x_1^{(k)},\ldots,x_n^{(k)} \in X$, and $k=1,...,m$. If $q<1$, the infimum of the constants C satisfying the above inequality is a Banach quasinorm for the space of absolutely $(q;1)$-summing m-linear operators from $X\times\cdots\times X$ to Y.

2. The proof of Theorem 1.1

Let $1 \leq g_m < \frac{2}{m+1}$. The first part of our argument is mentioned *en passant*, without proof, in [6], but since we have a more self-contained approach, we present the details for the sake of completeness. Let $n\in\mathbb{N}$ and $x_1,\ldots,x_n \in \ell_1$ be non null vectors. Consider $x_1^*,\ldots,x_n^* \in B_{\ell_\infty}$ so that $x_j^*(x_j) = \|x_j\|$ for every $j=1,\ldots,n$. Let $a_1,\ldots,a_n$ be scalars such that $\sum\limits_{j=1}^{n}|a_j|^{\frac{2}{g_m}} = 1$ and define

$$P_n\colon \ell_1 \longrightarrow \ell_2\ ,\ P_n(x) = \sum_{j=1}^{n}|a_j|^{\frac{1}{g_m}}\,x_j^*(x)^m e_j,$$

where e_j is the j-th canonical vector of ℓ_2. For every $x\in\ell_1$,

$$\|P_n(x)\| = \left(\sum_{j=1}^{n}\left||a_j|^{\frac{1}{g_m}}\,x_j^*(x)^m\right|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n}|a_j|^{\frac{2}{g_m}}\right)^{\frac{1}{2}}\|x\|^m = \|x\|^m.$$

Since P_n is a polynomial of finite type, then it is plain that P_n is absolutely $(g_m;1)$-summing. Note that for $k=1,\ldots,n$, we have

$$\|P_n(x_k)\| = \left\|\sum_{j=1}^{n}|a_j|^{\frac{1}{g_m}}\,x_j^*(x_k)^m e_j\right\| \geq |a_k|^{\frac{1}{g_m}}\,x_k^*(x_k)^m = |a_k|^{\frac{1}{g_m}}\,\|x_k\|^m.$$

To simplify the notation we write $\|(x_j)_{j=1}^n\|_{w,1} := \sup_{\varphi\in B_{X^*}}\sum_{j=1}^{n}|\varphi(x_j^{(k)})|$. Thus, we have

$$\begin{aligned}\left(\sum_{j=1}^{n}\|x_j\|^{mg_m}|a_j|\right)^{\frac{1}{g_m}} &= \left(\sum_{j=1}^{n}\left(\|x_j\|^m|a_j|^{\frac{1}{g_m}}\right)^{g_m}\right)^{\frac{1}{g_m}}\\ &\leq \left(\sum_{j=1}^{n}\|P_n(x_j)\|^{g_m}\right)^{\frac{1}{g_m}}\\ &\leq \pi_{g_m,1}(P_n)\|(x_j)_{j=1}^n\|_{w,1}^m.\end{aligned}$$

Since this last inequality holds whenever $\sum\limits_{j=1}^{n}|a_j|^{\frac{2}{g_m}}=1$, denoting $\left(\frac{2}{g_m}\right)^*$ to the conjugate of $\frac{2}{g_m}$ we obtain

$$\begin{aligned}\left(\sum_{j=1}^{n}\|x_j\|^{mg_m\left(\frac{2}{g_m}\right)^*}\right)^{\frac{1}{\left(\frac{2}{g_m}\right)^*}} &\leq \sup\left\{\sum_{j=1}^{n}|a_j|\,\|x_j\|^{mg_m};\sum_{j=1}^{n}|a_j|^{\frac{2}{g_m}}=1\right\}\\ &\leq \left(\pi_{g_m,1}(P_n)\|(x_j)_{j=1}^{n}\|_{w,1}^{m}\right)^{g_m}\end{aligned}$$

and, then,

$$\frac{\left(\sum\limits_{j=1}^{n}\|x_j\|^{mg_m\left(\frac{2}{g_m}\right)^*}\right)^{\frac{1}{g_m\left(\frac{2}{g_m}\right)^*}}}{\|(x_j)_{j=1}^{n}\|_{w,1}^{m}}\leq \pi_{g_m,1}(P_n). \tag{1}$$

Since $1\leq g_m<\frac{2}{m+1}$ we have $mg_m\left(\frac{2}{g_m}\right)^*<2$ and from a weak version of the *Dvoretzky-Rogers Theorem* we know that id_{ℓ_1} is not $\left(mg_m\left(\frac{2}{g_m}\right)^*;1\right)$-summing. Combining this fact with (1) we conclude that we can find x_j in ℓ_1 for all positive integer j so that

$$\lim_{n\to\infty}\pi_{g_m,1}(P_n)=\infty \text{ and } \|P_m\|=1. \tag{2}$$

We thus conclude that the space of all absolutely $(g_m;1)$-summing m-homogeneous polynomials from ℓ_1 to ℓ_2 is not closed in $\mathcal{P}\left({}^m\ell_1;\ell_2\right)$. In fact, otherwise, since the quasinorm $\pi_{g_m,1}$ is complete, the *Open Mapping Theorem* to F-spaces would contradict (2).

Now, let $P:\ell_1\to\ell_2$ be a continuous non $(g_m;1)$-summing m-homogeneous polynomial. Split $\mathbb{N}$ into a countable union of pairwise disjoint countable sets $\mathbb{N}_1,\mathbb{N}_2,\ldots$. For all j, let

$$\mathbb{N}_j=\left\{a_1^{(j)}<a_2^{(j)}<\cdots\right\},$$

and define $P^{(j)}:\ell_1\to\ell_2$ by $\left(P^{(j)}(x)\right)_{a_k^{(j)}}=(P(x))_k$ and $\left(P^{(j)}(x)\right)_k=0$ if $k\notin\mathbb{N}_j$. It is simple to prove that $P^{(j)}$ is also a continuous non $(g_m;1)$-summing m-homogeneous polynomial and the set $\left\{P^{(1)},P^{(2)},\ldots\right\}$ is linearly independent. Finally, we note that the linear operator $\Phi:\ell_1\to\mathcal{P}\left({}^n\ell_1;\ell_2\right)$ given by $(\beta_j)_{j=1}^{\infty}\mapsto\sum\limits_{j=1}^{\infty}\beta_jP^{(j)}$ is injective and it is simple to verify that $\Phi\left(\ell_1\right)$ is composed (except by the null vector) exclusively by non absolutely $(g_m;1)$-summing m-homogeneous polynomials. We thus conclude that the set of continuous non $(g_m;1)$-summing m-homogeneous polynomials is $\Phi\left(\ell_1\right)$-lineable. Since $\dim\Phi\left(\ell_1\right)=\dim\ell_1=\mathfrak{c}$ we conclude that this set is $\mathfrak{c}$-lineable.

Since $\mathcal{P}\left({}^n\ell_1;\ell_2\right)$ is isomorphic to the space of symmetric m-linear operators from $\ell_1\times\cdots\times\ell_1$ to ℓ_2 and since P is absolutely $(g_m;1)$-summing if and only if its associated symmetric m-linear operator is absolutely $(g_m;1)$-summing, our result is translated to the multilinear setting.

The above result is optimal in terms of dimension. In fact, it is well known that ℓ_1 is isometric to the completion of its projective tensor product, i.e., $\ell_1=\ell_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi\ell_1$. Thus

$$\dim\mathcal{L}\left({}^m\ell_1;\ell_2\right)=\dim\mathcal{L}\left(\ell_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi\ell_1;\ell_2\right)=\dim\mathcal{L}\left(\ell_1;\ell_2\right)=\mathfrak{c}.$$

We remark that if $\pi_{g_m,1}$ was locally convex, since we have proved that the space of all absolutely $(g_m;1)$-summing m-homogeneous polynomials from ℓ_1 to ℓ_2 is not closed in the space of all continuous m-homogeneous polynomials from ℓ_1 to ℓ_2, then from a result due to Drewnowski (see [8, Theorem 5.6 and its reformulation]) we would conclude that the set of all continuous m-homogeneous polynomials from ℓ_1 to ℓ_2 that fail to be absolutely $(g_m;1)$-summing is spaceable, i.e., contains (except for the null vector) a closed infinite-dimensional subspace.

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